§6 PSPACE and Polynomial Hierarchy

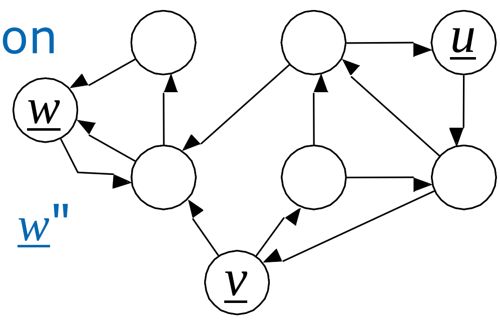
- $\mathcal{PSPACE}$-complete problems
  QBF, 3QBF, GRAPH
- Savitch's Theorem:
  $\mathcal{PSPACE} = \mathcal{NPSPACE}$
- Polynomial Hierarchy:
  syntactically / semanticall

Walter Savitch's Theorem

Def: For nondecreasing f:ℕ→ℕ, let TIME(f) :=
{ $L \subseteq$ {0,1}* decidable by WHILE+ program in time $f(n)$ }
Similarly NTIME(f), SPACE(f), NSPACE(f): nondet. WHILE+

Theorem: For any polynom. p, NSPACE(p)⊆SPACE(p^2)

Proof: Let nondet. $\mathcal{A}$ accept L in space s:=$p(n)$
Encode configurations of $\mathcal{A}$ in s bits $\underline{u}=(u_1,\ldots u_s)$. Draw edge from $\underline{u}$ to $\underline{v}$ if $\underline{v}$ encodes $\mathcal{A}$'s possible config after $\underline{u}$.
$\mathcal{G}=\mathcal{G}_{\mathcal{A},s}$ digraph with $\underline{u}$=start config on input x, $\underline{w}$ unique accepting config.
$\mathcal{A}$ accepts $x \Leftrightarrow P_{\mathcal{G}}(\underline{u},\underline{w},s)$,
:⇔ "∃path of length $\leq 2^s$ from $\underline{u}$ to $\underline{w}$"
$\Leftrightarrow \exists \underline{v}$: $P_{\mathcal{G}}(\underline{u},\underline{v},s\text{-}1) \wedge P_{\mathcal{G}}(\underline{v},\underline{w},s\text{-}1)$



Recursive algorithm of depth s stores $\underline{v}$ in s bits: O(s^2)

QBF and PSPACE

$\in$ co$\mathcal{NP}$?
$\in \mathcal{NP}$?
$\in \mathcal{PSPACE}$

QBF: Given Boolean term $\Phi(Y_1,\ldots Y_m)$, does it hold $\exists y_1\ \forall y_2\ \exists y_3\ \forall\ldots$: $\Phi(y_1,\ldots y_m)=1$?

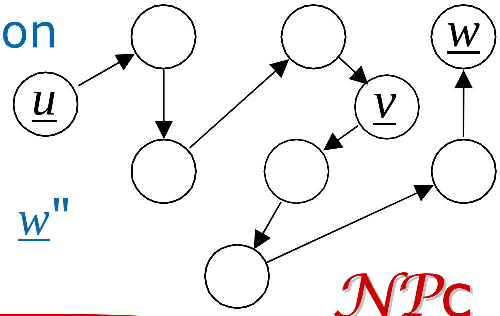
Recursively evaluate quantifiers: $s(m) = s(m\text{-}1) + \text{poly}(n)$

Theorem: **QBF** is $\mathcal{PSPACE}$-complete.

Proof: Let $\mathcal{A}$ decide $L\in$ **PSPACE** in space s:=poly(n).

Encode configurations of $\mathcal{A}$ in s bits $\underline{u}=(u_1,\ldots u_s)$. Draw edge from $\underline{u}$ to $\underline{v}$ if $\underline{v}$ encodes $\mathcal{A}$'s unique config after $\underline{u}$.

$\mathcal{G}=\mathcal{G}_{\mathcal{A},s}$ digraph with $\underline{u}$=start config on input x, $\underline{w}$ *unique* accepting config.



$\mathcal{A}$ accepts $x \Leftrightarrow P_{\mathcal{G}}(\underline{u},\underline{w},s)$,

:$\Leftrightarrow$ "$\exists$path of length $\leq 2^s$ from $\underline{u}$ to $\underline{w}$"

$\Leftrightarrow \exists\underline{v}$: $P_{\mathcal{G}}(\underline{u},\underline{v},s\text{-}1) \wedge P_{\mathcal{G}}(\underline{v},\underline{w},s\text{-}1)$

$\Leftrightarrow \exists\underline{v}\ \forall\underline{s},\underline{t}$: $(\underline{s}=\underline{u} \wedge \underline{t}=\underline{v}) \vee (\underline{s}=\underline{v} \wedge \underline{t}=\underline{w}) \to P_{\mathcal{G}}(\underline{s},\underline{t},s\text{-}1)$

$\mathcal{NP}$c

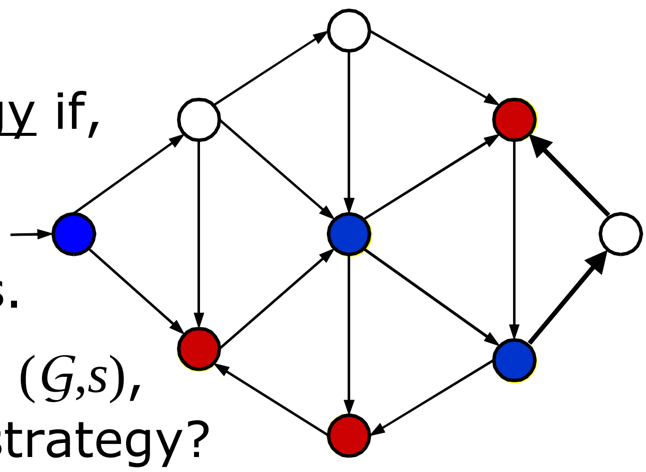
Two-Player Game on Graphs

3QBF: Given Boolean term $\Phi(X_1,\ldots X_m)$ in 3CNF, does it hold $\exists x_1\ \forall x_2\ \exists x_3\ \forall\ldots$: $\Phi(x_1,\ldots x_m)=1$? $\mathcal{PSPACE}$c

Fix digraph $\mathcal{G}$ with start vertex s. **Rules:**

- **Red** (start) and **blue** player alternatingly
- mark current vertex, and follow any outgoing edge
- to a yet unmarked vertex.
- Who cannot move, loses.

Red has a winning strategy if, however **blue** reacts, **red** can follow such that, however ……… **blue** looses.



Decision problem: Given ($\mathcal{G}$,s), does **red** have a winning strategy?

Two-Player Game on Graphs

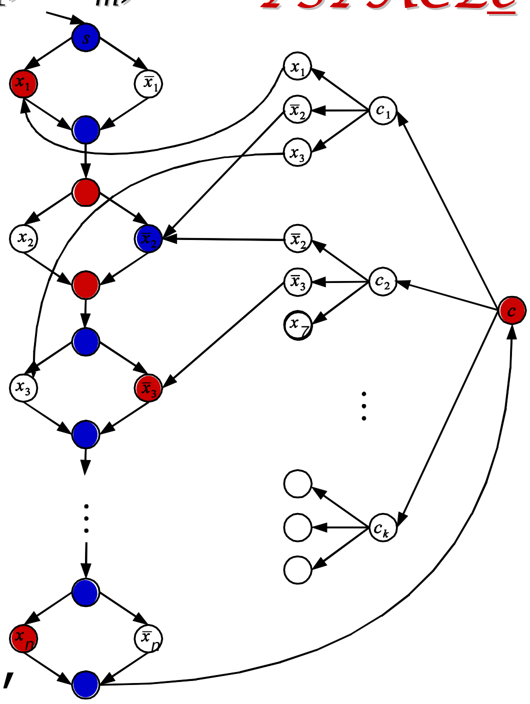
3QBF: Given Boolean term $\Phi(X_1,\ldots X_m)$ in 3CNF, does it hold $\exists x_1\, \forall x_2\, \exists x_3\, \forall\ldots : \Phi(x_1,\ldots x_m)=1$? $\mathcal{PSPACE}$c

Proof (reduction from **3QBF**):

Let $\Phi = C_1 \wedge C_2 \wedge \ldots \wedge C_k$

See illustration for

$\Phi = (y_1 \vee \neg y_2 \vee y_3)$
$\wedge (\neg y_2 \vee \neg y_3 \vee y_7)$
$\wedge \ldots \wedge C_k$



Theorem: The following is $\mathcal{PSPACE}$-complete:

Decision problem: Given $(\mathcal{G},s)$, does **red** have a winning strategy?

Oracle Complexity Theory

Fix $O\subseteq\mathbb{N}$. **OWHILE+** progr. $\mathcal{A}^O$ has test instrc. "$x_j\in O$**?**"

Definition: Fix some class $\mathbf{C}$ of subsets $L\subseteq\mathbb{N}$.

$\mathbf{P}^{\mathbf{C}}$:= { $L\subseteq\mathbb{N}$ decided by polytime **OWHILE+** $\mathcal{A}^O$, $O\in\mathbf{C}$ }

$\mathbf{NP}^{\mathbf{C}}$:= { $L\subseteq\mathbb{N}$ accep. nondet.poly. **OWHILE+** $\mathcal{A}^O$, $O\in\mathbf{C}$ }

Examples:

a) $\mathbf{MinBF} \in \mathbf{NP}^{\mathrm{SAT}} = \mathbf{NP}^{\mathbf{NP}} \subseteq \mathbf{P}^{\mathbf{NP}^{\mathbf{NP}}}$ (see homework problem 11e)

b) $\mathbf{P}^{\mathbf{P}}=\mathbf{P}$, $\mathbf{NP}^{\mathbf{P}}=\mathbf{NP}$, $\mathbf{PSPACE}^{\mathbf{PSPACE}}=\mathbf{PSPACE}$

c) $\mathbf{NP} \cup \mathbf{coNP} \subseteq \mathbf{P}^{\mathbf{NP}}$; „$\neq$" unless $\mathbf{NP}=\mathbf{coNP}$

Semantic Polynomial Hierarchy

Def: $\Delta_0\mathbf{P} = \Sigma_0\mathbf{P} = \Pi_0\mathbf{P} := \mathbf{P}$

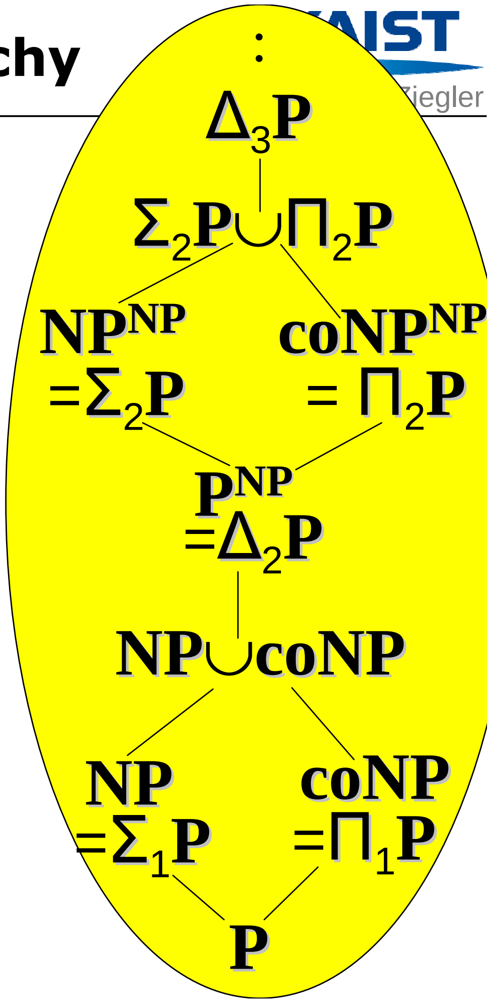
- $\Delta_{k+1}\mathbf{P} := \mathbf{P}^{\Sigma_k\mathbf{P}} = \mathbf{P}^{\Pi_k\mathbf{P}}$
- $\Sigma_{k+1}\mathbf{P} := \mathbf{NP}^{\Sigma_k\mathbf{P}} = \mathbf{NP}^{\Pi_k\mathbf{P}}$
- $\Pi_{k+1}\mathbf{P} := \mathbf{coNP}^{\Sigma_k\mathbf{P}} = \mathbf{coNP}^{\Pi_k\mathbf{P}}$
- $\mathrm{PH} := \bigcup \Sigma_k\mathbf{P}$

Lemma: a) $\Delta_k\mathbf{P} = \text{co-}\Delta_k\mathbf{P}$

b) $\Delta_k\mathbf{P} \subseteq \Sigma_k\mathbf{P} \cap \Pi_k\mathbf{P}$

c) $\Sigma_k\mathbf{P} \cup \Pi_k\mathbf{P} \subseteq \Delta_{k+1}\mathbf{P}$

d) $\mathrm{PH} \subseteq \mathbf{PSPACE}$



Syntactic Polynomial Hierarchy

Abbreviate $\mathbb{N}_n := \{\, y\in\mathbb{N} : \ell(y)\leq n \,\}$

Theorem: a) $L\subseteq\mathbb{N}$ belongs to $\mathbf{NP}$ iff

$L = \{\, x \mid \exists y\in\mathbb{N}_{\mathrm{poly}(n)} : \langle x,y\rangle\in V \,\}$ for some $V\in\mathbf{P}$ ($n:=\ell(x)$)

b) L belongs to Σ_{k+1} iff

$L = \{\, x \mid \exists y\in\mathbb{N}_{\mathrm{poly}(n)} : \langle x,y\rangle\in W \,\}$ for some $W\in\Pi_k$ or Σ_k

c) L belongs to Π_{k+1} iff

$L = \{\, x \mid \forall y\in\mathbb{N}_{\mathrm{poly}(n)} : \langle x,y\rangle\in Z \,\}$ for some $Z\in\Sigma_k$ or Π_k

d) L belongs to Σ_k iff

$L = \{\, x \mid \exists y_1\in\mathbb{N}_{\mathrm{poly}(n)}\ \forall y_2\in\mathbb{N}_{\mathrm{poly}(n)}\ \exists y_3\ldots$
$Q y_k\in\mathbb{N}_{\mathrm{poly}(n)} : \langle x,y_1,y_2,\ldots y_k\rangle\in A \,\}$ for some $A\in\mathbf{P}$

("∃" if k odd, "∀" else)

$\Sigma_{k+1}\mathbf{P} = \mathbf{NP}^{\Sigma_k\mathbf{P}} = \mathbf{NP}^{\Pi_k\mathbf{P}}$ $\quad \Pi_{k+1}\mathbf{P} = \mathbf{coNP}^{\Sigma_k\mathbf{P}} = \mathbf{coNP}^{\Pi_k\mathbf{P}}$