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CS422

Fall 2017, Assignment #4

PROBLEM 8 (2+2P) **:**

a) Prove that $\mathcal{P}$ is closed under

   i) binary union, i.e., $A,B \in \mathcal{P} \Rightarrow A \cup B \in \mathcal{P}$
   ii) intersection, i.e. $A,B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$
   iii) complement, i.e. $A \in \mathcal{P} \Rightarrow \mathbb{N} \setminus A \in \mathcal{P}$
   iv) product, i.e. $A,B \in \mathcal{P} \Rightarrow A \times B := \{\langle a,b\rangle : a \in A, b \in B\} \in \mathcal{P}$.

b) Prove that also $\mathcal{NP}$ is closed under i) union, ii) intersection, ~~iii) complement,~~ iv) product. Give *two* proofs for each: One based on non-deterministic `WHILE+` programs only, and one based on problems with polynomial-time verifiable witnesses. What makes (iii) difficult?

PROBLEM 9 (1+2P) **:**

a) Use the Boolean connectives $\vee, \wedge, \neg$ to construct formulae $\psi(x,y,z)$ and $\chi(x,y,z)$ such that $\psi + 2\chi$ is the binary expansion of $x+y+z$ for all $x,y,z \in \{0,1\}$.

b) Construct a Boolean formula $\varphi_n(\vec{x},\vec{y},\vec{z},\vec{c})$ with the following property:
For $\vec{x},\vec{y},\vec{z} \in \{0,1\}^n$, $\mathrm{bin}(\vec{z}) = z_0 + 2z_1 + \ldots + 2^{n-1}z_{n-1}$ is the binary expansion of the sum $\mathrm{bin}(\vec{x}) + \mathrm{bin}(\vec{y})$ iff there exists a $\vec{c} \in \{0,1\}^n$ such that $\varphi_n(\vec{x},\vec{y},\vec{z},\vec{c}) = 1$.

PROBLEM 10 (2+2+2P) **:**
Recall the problem *Integer Linear Programming* from the lecture:

$$\mathsf{ILP} \;=\; \big\{\langle A,\vec{b}\rangle \;\big|\; n,m \in \mathbb{N},; A \in \mathbb{Z}^{n\times m},\ \vec{b} \in \mathbb{Z}^m,\ \exists \vec{x} \in \mathbb{N}^n:\ A \cdot \vec{x} = \vec{b}\big\}$$

a) Devise a polynomial-time reduction from the Boolean satisfiability problem, SAT, to ILP.

b) Devise a polynomial-time reduction from the following problem $\mathsf{SubsetSum}$ to ILP:

$$\big\{\langle v_1,\ldots,v_N,v_0\rangle\ :\ N,v_1,\ldots,v_N,v_0 \in \mathbb{N},\ \exists b_1,\ldots,b_N \in \{0,1\}:\ b_1v_1 + \cdots + b_Nv_N = v_0\big\}$$

c) Use Problem 9 to describe a polynomial-time reduction from $\mathsf{SubsetSum}$ to SAT.