„Introduction to Algorithms“

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Syllabus

3. Sorting

- Specification
- Bubble Sort
- Selection Sort
- Insertion Sort
- Merge Sort
- Quicksort
- Linear-Time Median
- Sorting in Linear Time
- Sorting in Parallel

- Specification
- Primitives:
  semantics and cost
- Design
- Analysis
- Optimality

of Comparison-Based Sorting

3. Sorting

specification

Specification: Fix set X with total order $\leq$.

Input: $N \in \mathbb{N}$ and finite sequence $x_1, \ldots x_N \in X$ in array $x[1 \ldots N]$

and array $\pi[1 \ldots N]$:= identity permutation of $\{1, \ldots N\}$

Output: Permutation π of $\{1, \ldots N\}$ such that $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives: ordered comparison "$x[n] \leq x[m]$?" cost 1

Index integer arithmetic cost 1

swapping $x[n] \leftrightarrow x[m]$ cost 1

swapping $\pi[n] \leftrightarrow \pi[m]$ cost 1

Preprocessing data in order to accelerate queries.

3. Sorting

Bubble Sort

Input: $N \in \mathbb{N}$ and array $x[1\ldots N]$

Output: *Permuted* array $x \circ \pi$
s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

```
Procedure BubbleSort ( $x[N]$ )
For  $m := N$  downto 2  do
    For  $k := 1$ to $m-1$ do   ←
        If  $x[k] > x[k+1]$  then
            Swap ($x[k]$ , $x[k+1]$)
        Endif
    Endfor
Endfor
```

Primitives:

Comparison "$x[n] \leq x[m]$?"

Swapping $x[n] \leftrightarrow x[m]$

Index integer arithmetic

runtime $O(N^2)$

Correctness: $x[1], \ldots x[m] \leq x[m+1] \leq \ldots \leq x[N]$

3. Sorting

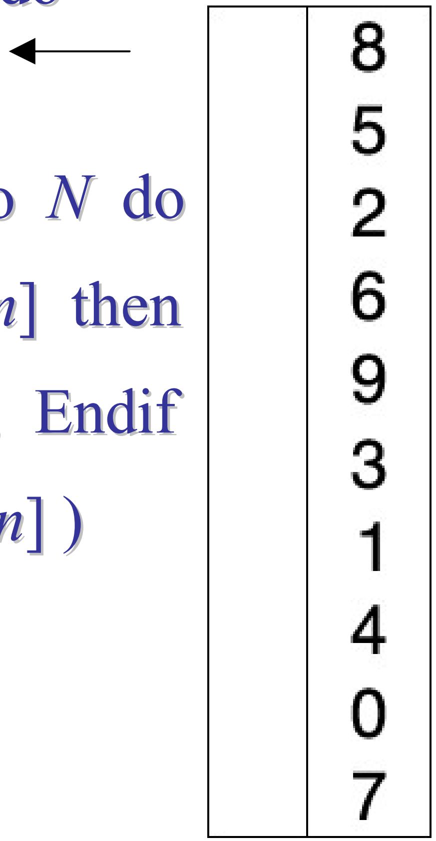
Select Sort

Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$

Output: *Permuted* array $x \circ \pi$

s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

```
Procedure SelectSort ( $x[N]$ )
For $m := 1$ to $N-1$ do
    $min := m$;
    For $k := min+1$ to $N$ do
        If $x[k] < x[min]$ then
            $min := k$ ; Endif
    Swap( $x[m], x[min]$ )
    Endfor
Endfor
```



Primitives:

Comparison "$x[n] \leq x[m]$?"

Swapping $x[n] \leftrightarrow x[m]$

Index integer arithmetic

runtime $O(N^2)$

Correctness: $x[1] \leq \ldots \leq x[m-1] \leq x[m], \ldots x[N]$

3. Sorting

Insert Sort

6 5 3 1 8 7 2 4

```
Procedure **InsertSort** ( $x[N]$ )
For $m := 2$ to $N$ do   ←
    $y := x[m];$   $k := m-1;$
    While $k>0$ and $x[k]>y$
        $x[k+1] := x[k]$
        $k := k-1$
    Endwhile
    $x[k+1] := y$
Endfor
```

Input: $N \in \mathbb{N}$ and array $x[1\ldots N]$

Output: *Permuted* array $x \circ \pi$

s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives:

Comparison "$x[n] \leq x[m]$?"

Swapping $x[n] \leftrightarrow x[m]$

Index integer arithmetic

runtime $O(N^2)$

Correctness: $x[1] \leq \ldots \leq x[m-1] \leq x[m], \ldots x[N]$

3. Sorting

Merge Sort

Input: $N \in \mathbb{N}$ and array $x[1\ldots N]$

Output: *Permuted* array $x \circ \pi$
s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

```
Procedure MergeSort ( $x[N]$ )
If $N \leq 1$ return.
$l := \lfloor N/2 \rfloor$; $r := \lceil N/2 \rceil$; array $y[l]$, $z[r]$;
For $m := 1$ to $l$ do $y[m] := x[m]$;
For $m := 1$ to $r$ do $z[m] := x[l+m]$;
MergeSort($y$);  MergeSort($z$);
While $l>0$ and $r>0$ do ; If $z[r] < y[l]$
        then  $x[l+r] := y[l]$ ; $l := l-1$
        else  $x[l+r] := z[r]$ ; $r := r-1$
While $l>0$ do ; $x[l+r] := y[l]$ ; $l := l-1$ ; Endwhile
While $r>0$ do ; $x[l+r] := z[r]$ ; $r := r-1$ ; Endwhile
```

6 5 3 1 8 7 2 4

runtime

$$T(N) = 2 \cdot T(N/2) + O(N)$$
$$\leq O(N \cdot \log N)$$

memory

$$O(N \cdot \log N)$$

3. Sorting

Quick Sort

Input: $N \in \mathbb{N}$ and array $x[1\ldots N]$

Output: *Permuted* array $x \circ \pi$
s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

```
Procedure QuickSort ( $x[]$ ; $l,r$ )
If $l \geq r$ return.   // $x[l] \ldots x[r]$ sorted
$s := x[\lfloor (l+r)/2 \rfloor] \in X$   // sample pivot
$a := l$ ; $b := r$ ; While $a < b$ do
    While $a<b$ and $x[a] < s$ do $a := a+1$ Endwhile;
    While $a<b$ and $x[b] \geq s$ do $b := b-1$ Endwhile;
    Swap($x[a]$,$x[b]$);
    Endwhile ;   // $l \leq a = b \leq r$
QuickSort ( $x$ , $l$ , $a-1$);
QuickSort ( $x$ , $b+1$ , $r$);
```

$$T(N) \leq O(N) + \\ + T(N\text{-}2) + T(2) \\ = O(N^2)$$

l $a{=}b$ r

Idea: $s := x[p]$ for ***random*** $p \in [l..r]$

QuickSort is in worst-case as bad as **`BubbleSort`**

→ *randomized* algorithms

3. Sorting

continued: Quick Sort

Input: $N\in\mathbb{N}$ and array $x[1\ldots N]$

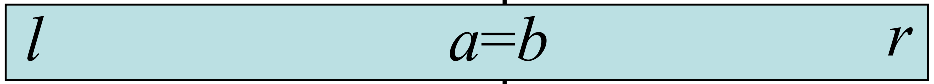
Output: *Permuted* array $x\circ\pi$ s.t. $x_{\pi(1)}\leq\ldots\leq x_{\pi(N)}$

Procedure **QuickSort** ($x[]$; l,r)

```
If $l \geq r$ return.   // $x[l]\ldots x[r]$ sorted
$s := x[\lfloor(l+r)/2\rfloor] \in X$   // sample pivot
$a := l$ ;  $b := r$ ;  While  $a < b$  do
    While  $a<b$  and  $x[a] < y$  do  $a := a+1$   Endwhile;
    While  $a<b$  and  $x[b] \geq y$  do  $b := b-1$   Endwhile;
    Swap($x[a],x[b]$);
    Endwhile ;     Return $a$;
QuickSort ( $x$ , $l$ , $a-1$);
QuickSort ( $x$ , $b+1$ , $r$);
```

$$T(N) = T(N\cdot\varepsilon) + T(N\cdot(1-\varepsilon)) + O(N)$$

$\leq O(N\cdot\log N)$ if $\varepsilon\in(0,1)$ ***fixed***



$$\varepsilon\cdot(r\text{-}l+1) \leq a-l \leq (1\text{-}\varepsilon)\cdot(r\text{-}l+1)$$

$$\varepsilon\cdot(r\text{-}l+1) \leq r-a \leq (1\text{-}\varepsilon)\cdot(r\text{-}l+1)$$

$$c\cdot N\cdot\log(N) =: T(N) = c\cdot N\cdot\varepsilon\cdot\log(N\cdot\varepsilon) + c\cdot N\cdot(1\text{-}\varepsilon)\cdot\log(N\cdot(1\text{-}\varepsilon)) + N$$

Ansatz

$$= c\cdot N\cdot\log(N) + N\cdot c\cdot\boxed{(\varepsilon\cdot\log(\varepsilon) + (1\text{-}\varepsilon)\cdot\log(1\text{-}\varepsilon))} + N$$

3. Sorting

Median Revisited

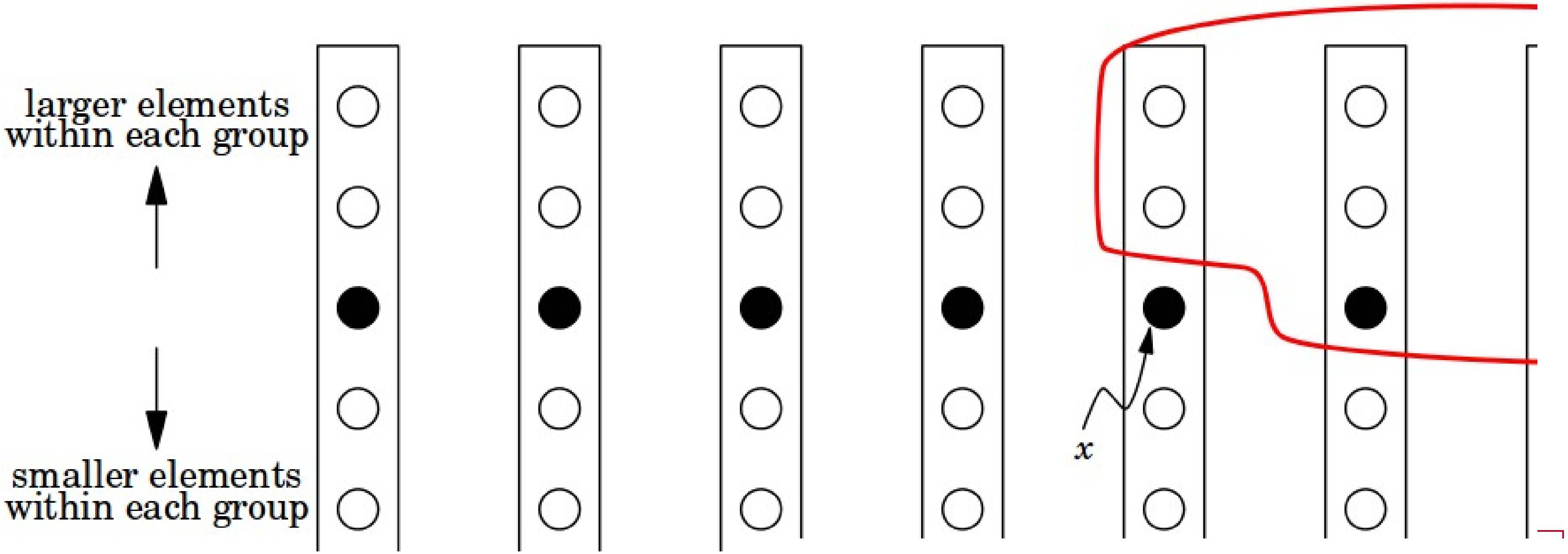
Function **Partition** $(x[]\ ;\ l\ ,\ r\ ;\ s)$

// splits $x[l..r]$: into $x[l..m]$ with entries $<s$

// and $x[m+1..r]$ those $\geq s$. Returns m.

Input: $N\in\mathbb{N}$ and array $x[1\ldots N]$

W.l.o.g. $x[n]\neq x[m]$ for all $n\neq m$



K-th order statistic: m s.t.

$\#\{n : x_n<x_m\} < K \leq \#\{n : x_n\leq x_m\}$

Lemma: *Median of 5-medians* is "$\leq$" at least $\frac{1}{2}\cdot\frac{3}{5} = 30\%$ of entries, and "**>**" at most 70%.

Output: m such that $0.3\cdot N \leq \#\{n : x_n \leq x_m \} \leq 0.7\cdot N+1$

3. Sorting

Linear-time Median

```
Function Partition ( $x[]$ ; $l$ , $r$ ; $y$)
// partitions $x[l..r]$ into $x[l..m]$ entries $<y$
// and $x[m+1..r]$ those $\geq y$. Returns $m$.
```

Input: $N\in\mathbb{N}$ and array $x[1\ldots N]$

W.l.o.g. $x[n]\neq x[m]$ for all $n\neq m$

```
Function OrderStat ( $x[]$ ; $l$, $r$ , $K$)
While $l<r$ do
    Call ApproxMed in order to
        determine an approximate
        median $s$ of $x[l\ldots r]$.
    Partition $x[l\ldots r]$ accordingly.
    Proceed to the left (=decrease $r$)
    /right (=increase $l$) accordingly.
```

```
Function ApproxMed ( $x[]$ ; $l$ , $r$)
Process $x[l...r]$ in groups of 5:
    For each group, find its median
    (for instance by brute force).
Then call OrderStat to determine
the median of these 5medians.
```

K-th order statistic: m s.t. $\#\{n : x_n<x_m\} < K \leq \#\{n : x_n\leq x_m\}$

$n := r - l+1$

Lemma: *Median of 5-medians* is "$\leq$" at least $\frac{1}{2}\cdot\frac{3}{5} = 30\%$ of entries, and ">" at most 70%.

$$T_A(n) = O(n) + T_O(0.2\cdot n) , \qquad T_O(n) = T_A(n) + O(n) + T_O(0.7\cdot n)$$

3. Sorting

Optimality

Specification: Fix set X with total order $\leq$.

Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$ with values in X

Output: *Permutation* π s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives

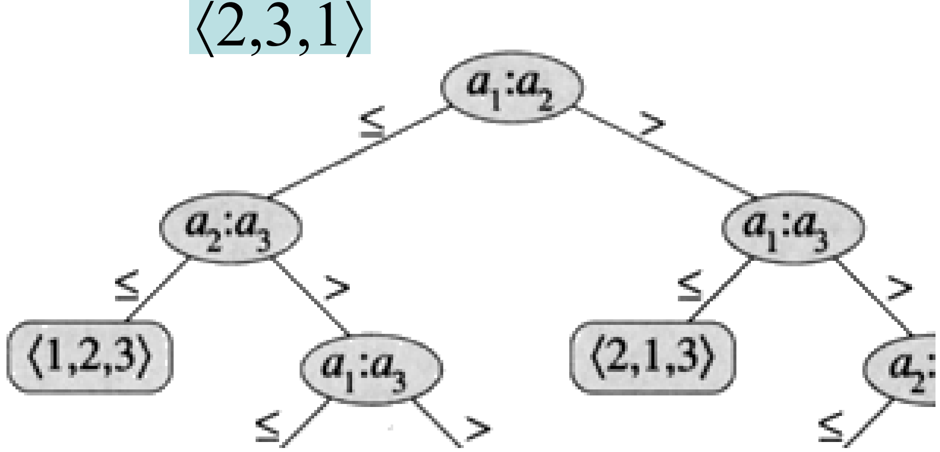
"$x[\pi[n]] \leq x[\pi[m]]$?"

Swap $\pi[n] \leftrightarrow \pi[m]$

Index arithmetic

Definition: A *Decision Tree for sorting N elements* is a binary tree whose nodes are labeled "$x[i] \leq x[j]$?", $1 \leq i < j \leq N$.

and whose leaves are labeled with permutations π such that every input $x = x[1 \ldots N]$ ends up in a leaf whose label π satisfies $x[\pi[1]] \leq \ldots \leq x[\pi[N]]$.



3. Sorting

Optimality

Lemma: a) For fixed N, every *comparison-based* sorting algorithm of worst-case runtime $T(N)$ can be “unrolled” into a decision tree for sorting N elements of depth $\leq T(N)$.

b) Every permutation $x=\pi$ ends up in the unique leaf with label π^{-1}.

c) A binary tree with $N!$ leaves has depth $\geq \log(N!)=\Theta(N\cdot\log N)$

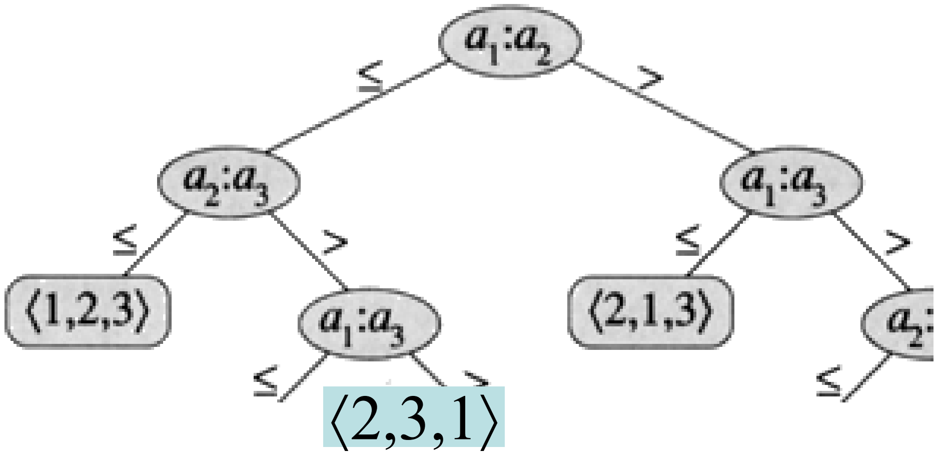
Definition: A *Decision Tree for sorting N elements* is a binary tree whose internal nodes are labeled “$x[i]\leq x[j]$?”, $1\leq i<j\leq N$.

and whose leaves are labeled with permutations π.

“$x[\pi[n]]\leq x[\pi[m]]$?”

Swap $\pi[n] \leftrightarrow \pi[m]$

Index arithmetic



Theorem: Any comparison based sorting algorithm requires time at least $\Omega(N\cdot\log N)$.

3. Sorting

Counting Sort

Specification: Fix set $X=\{1,\dots M\}$!

Input: $N\in\mathbb{N}$ and array $x[1\dots N]$ with key values in X

Output: *Permuted* array $y=x\circ\pi$ s.t. $x.key[\pi[1]]\leq\dots\leq x.key[\pi[N]]$

```
integer array $count[1\dots M]$;
For $m$:=1 to $M$ do $count[m] := 0$;
For $n$:=1 to $N$ do $count[x.key[n]]$++;
For $m$:=2 to $M$ do
  $count[m]$ += $count[m-1]$;
Endfor;
For $n$:=1 to $N$ do
  $y[count[x[n].key]$--$] := x[n]$;
```

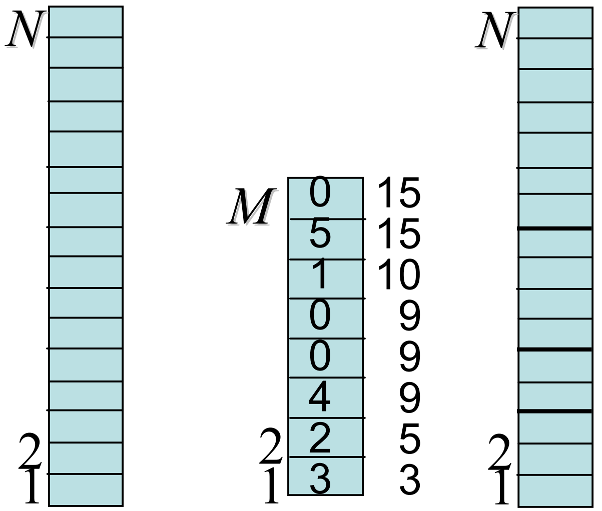
runtime $O(N+M)$

memory $O(N+M)$

~~“$x[\pi[n]]\leq x[\pi[m]]$?”~~

~~Swap $\pi[n]\leftrightarrow\pi[m]$~~

~~Index arithmetic~~



3. Sorting

Radix Sort

Specification: Fix set $X=\{0,1\}^m$ with lexicographical order.

$0\ b_{m-1} \dots b_2\ b_1 < 1\ c_{m-1} \dots c_2\ c_1$

Input: $N\in\mathbb{N}$ and array $x[1\dots N]$ with values in X

Output: *Permuted* array $y=x\circ\pi$ s.t. $y[1]\leq\dots\leq y[N]$

Quick-/Mergesort
Bit-cost: $O(N\cdot\log N\cdot m)$

~~"$x[n]\leq x[m]$?"~~

~~Swap $x[n]\leftrightarrow x[m]$~~

Index arithmetic

```
RadixSort( $x[]$, $l$, $r$, $m$ );
// Sort $x[l\dots r]$ w.r.t. bits #$m$…#1
If $l=r$ or $m<1$ then return;
// Put all entries with 0 as bit #$m$ before those with 1 :
$mid$ := Partition ( $x[]$ , $l$ , $r$ , $m$);
RadixSort( $x[]$ , $l$ , $mid$ , $m$-1);
RadixSort( $x[]$ , $mid$+1 , $r$ , $m$-1);
```

$O(N\cdot m)$ operations

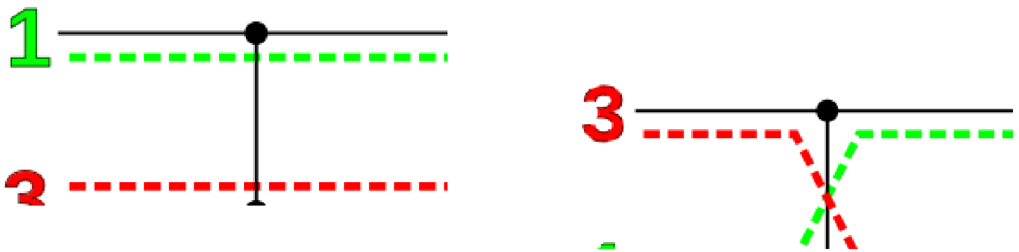
Bit-cost: $O(N\cdot m^2)$

3. Sorting *in Parallel* Sorting Networks

Specification: Fix set X with total order $\le$.

Input: $N\in\mathbb{N}$ and values $x[1\ldots N]$ in X

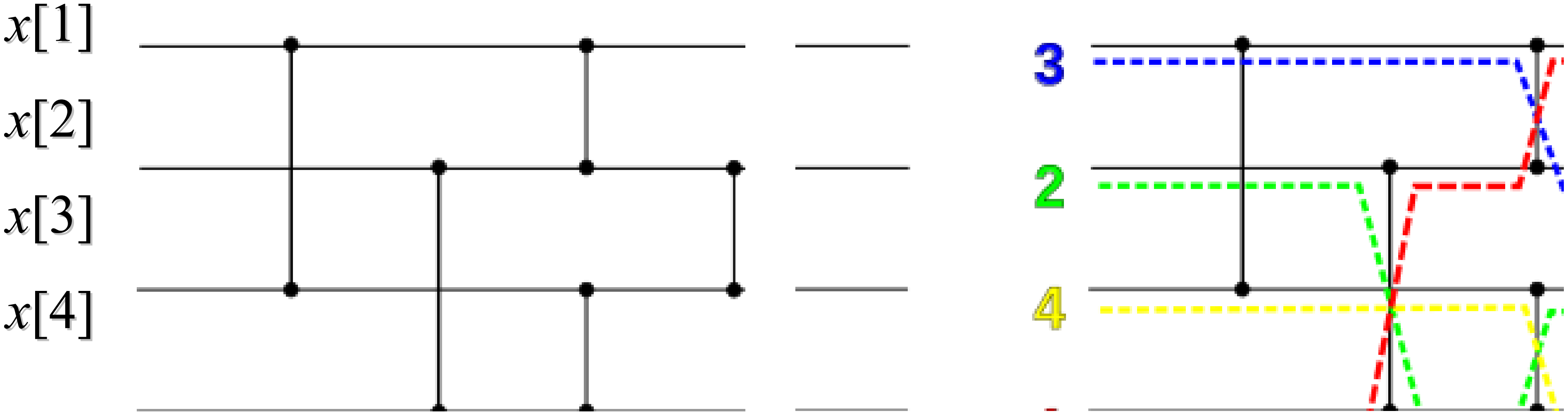
Output: *Permuted* array $y=x\circ\pi$ s.t. $x[\pi[1]]\le\ldots\le x[\pi[N]]$



One primitive Gate

"If $x[n]\le x[m]$ then swap $x[n]\leftrightarrow x[m]$"

$N=4$:



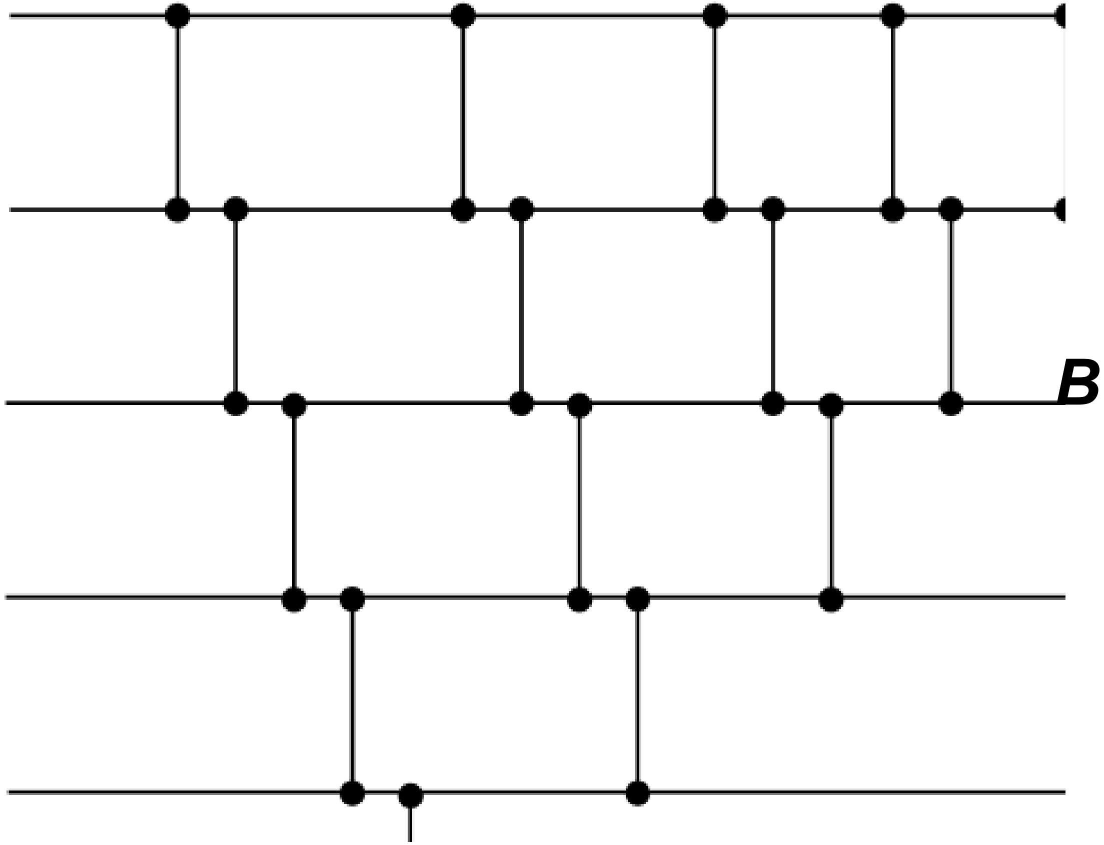
3. Sorting *in Parallel*

Sorting Networks (continued)

Specification: Fix set X with total order $\leq$.

Input: $N \in \mathbb{N}$ and values $x[1 \ldots N]$ in X

Output: *Permuted* array $y = x \circ \pi$ s.t. $x[\pi[1]] \leq \ldots \leq x[\pi[N]]$



One primitive Gate

"If $x[n] \leq x[m]$ then

swap $x[n] \leftrightarrow x[m]$"

***Bubble* Sorting Network:**

Size $O(N^2)$,

Depth (=parallel time) $O(N)$

3. Sorting *in Parallel*

Sorting Networks (continued)

Specification: Fix set $\boldsymbol{X}$ with total order $\leq$.

Input: $N \in \mathbb{N}$ and values $x[1 \ldots N]$ in X

Output: *Permuted* array $\boldsymbol{y} = x \circ \pi$ s.t. $x[\pi[1]] \leq \ldots \leq x[\pi[N]]$

One primitive Gate

"If $x[n] \leq x[m]$ then swap $x[n] \leftrightarrow x[m]$"

Theorem (without proof):

a) If a sorting network correctly sorts all sequences $x[1 \ldots N]$ with **values in {0,1}**,

then it correctly sorts all sequences with **values in $\boldsymbol{X}$**.

b) There exist sorting networks of size $O(N \cdot \log N)$ and depth $O(\log N)$.

This is optimal

***Bubble* Sorting Network:**

Size $O(N^2)$,

Depth (=parallel time) $O(N)$

Summary

3. Sorting

- Specification
- Bubble Sort
- Selection Sort
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- Merge Sort
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- Sorting in Parallel

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of Comparison-Based Sorting