§7 Complexity Theory

- Complexity Classes $\mathcal{P}$ and $\mathcal{NP}$
- Eulerian/Hamiltonian Cycle
- Edge/Vertex Cover
- Clique, Independent Set
- Comparing Decision Problems
- Travelling Salesperson
- Knapsack

Complexity Classes $\mathcal{P}$, $\mathcal{NP}$

Definition: A decision problem $L \subseteq \{0,1\}^*$

is decidable in polynomial time if, given $\underline{x} \in \{0,1\}^n$, $|\underline{x}|=n$, ...

- $\mathcal{P} = \{$ L decidable in polynomial time $\}$
- $\mathcal{NP} = \{$ L verifiable in polynomial time $\}$, i.e.

$$L = \{ \underline{x} : \exists \underline{y}: |\underline{y}| \leq \text{poly}(|\underline{x}|),\ \langle \underline{x},\underline{y} \rangle \in V \},\quad V \in \mathcal{P}$$

- $\mathcal{EXP} = \{$ L decidable in exponential time $\}$

Example: $L = \{0,1\}^* \setminus \{$**10, 11, 101, 111, 1011, 1101**, ...$\}$

$\mathcal{NP}$ $= \{ \underline{x}: \exists \underline{y}: |\underline{y}| < |\underline{x}|,\ 1 < \text{bin}(\underline{y}) \mid \text{bin}(\underline{x}) \}$

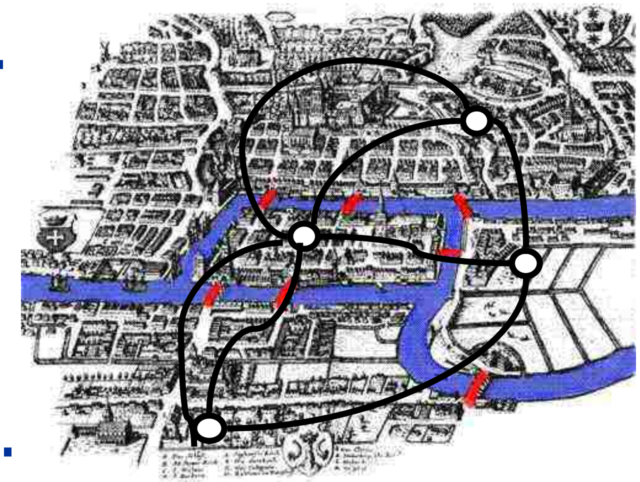
$$L = \{ \underline{x} : \exists \underline{y}: |\underline{y}| \leq \text{poly}(|\underline{x}|),\ \langle \underline{x},\underline{y} \rangle \in V \},\quad V \in \mathcal{P}$$

Eulerian/Hamiltonian Cycle

G undirected (multi)graph.

A Eulerian cycle traverses each edge precisely once;

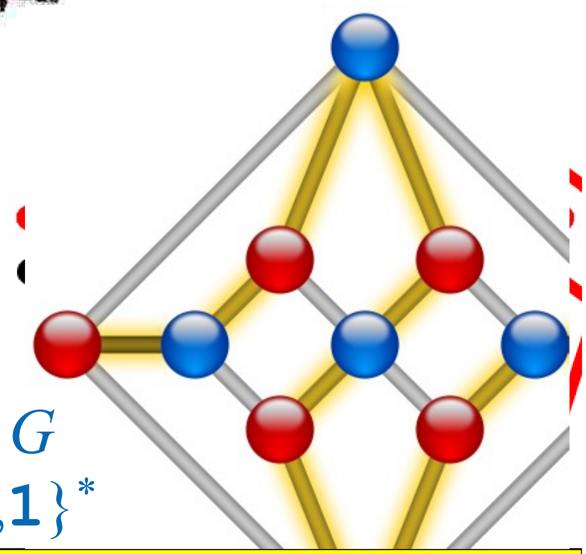
A Hamiltonian cycle visits each vertex precisely once.



Lemma: Graph G without isolated vertices   admits a Eulerian cycle

iff (i) G is connected and

(ii) each vertex has even degree.



encode G over $\{0,1\}^*$

EC := { ⟨G⟩ | G has a Eulerian cycle} $\mathcal{NP}$

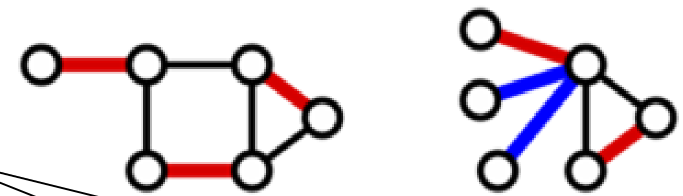
HC := { ⟨G⟩ | G has Hamiltonian cycle} $\mathcal{NP}$

$\mathcal{NP} \ni L = \{ \underline{x} : \exists \underline{y}: |\underline{y}| \le \mathrm{poly}(|\underline{x}|),\ \langle \underline{x},\underline{y}\rangle \in V \},\quad V \in \mathcal{P}$

Edge/Vertex Cover, Clique, IS

- Eulerian **(EC,$\mathcal{P}$)** vs. Hamiltonian Cycle **(HC,$\mathcal{NP}$)**

- (Minimum) **Edge Cover** $\mathcal{NP}$

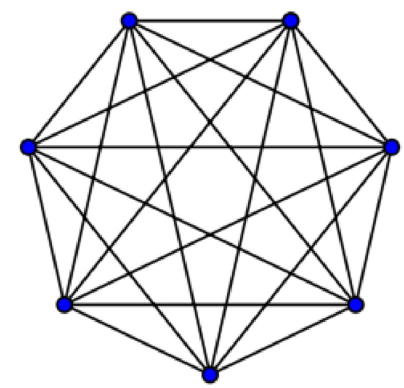


Greedily extend a maximum matching (Edmonds' *Blossom*)

- (min) **Vertex Cover (VC)** $\mathcal{NP}$

- **CLIQUE** = $\{ \langle G,k \rangle \mid$
  G contains a k-clique $\}$

$\mathcal{NP}$ **?**



- **IS**=$\{\langle G,k \rangle$:
  G has k pairwise *non*-adjacent vertices$\}$

Optimization ↔ Decision

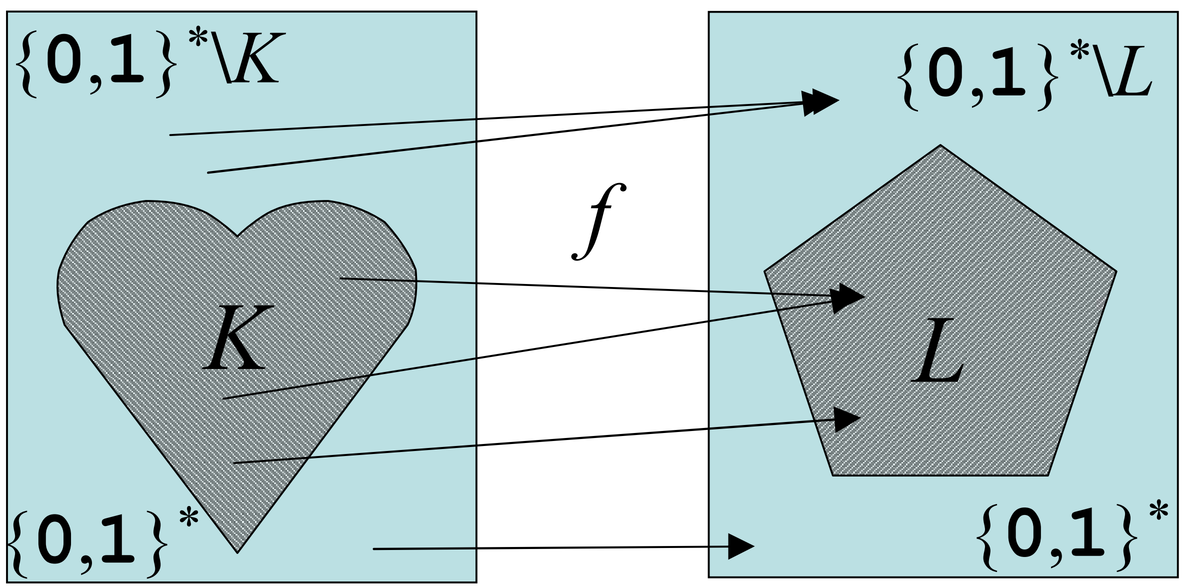
$$\mathbf{VC} = \{\langle V,E,k \rangle : \exists U \subseteq V,\ |U|=k,\ \forall (x,y) \in E:\ x \in U \vee y \in U \}$$

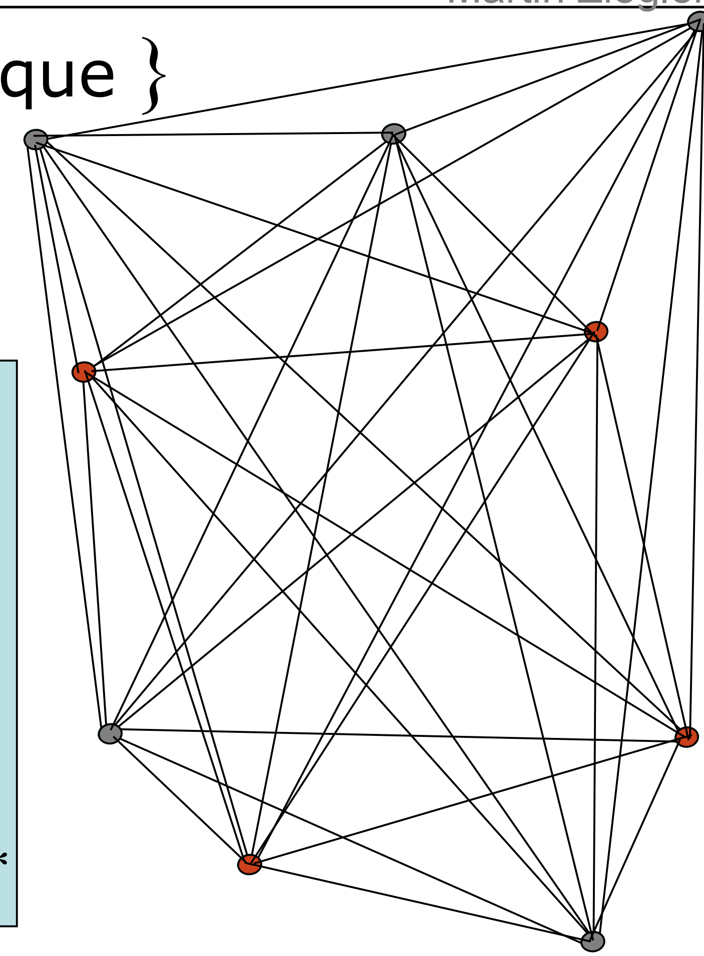
$$\mathcal{NP} \ni L = \{ \underline{x} : \exists \underline{y}: |\underline{y}| \leq \mathrm{poly}(|\underline{x}|),\ \langle \underline{x},\underline{y} \rangle \in V \},\quad V \in \mathcal{P}$$

Comparing Decision Problems

CLIQUE = { ⟨G,k⟩ | G contains a k-clique }

$\equiv_p$ **IS**={ ⟨G,k⟩ : G has k pairwise *non*-adjacent vertices}





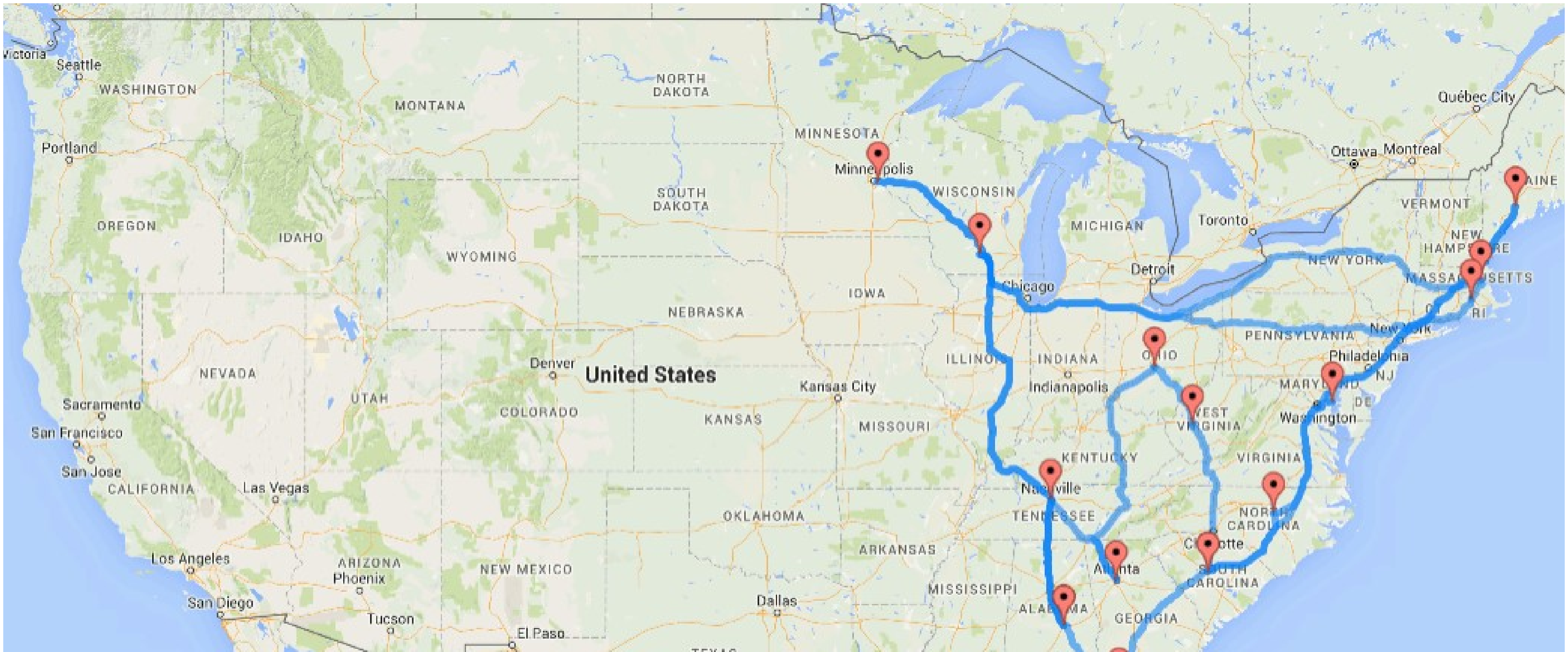
For $K,L \subseteq \{0,1\}^*$ write $K \preccurlyeq_p L$ if, for some polynomial-time computable $f: \{0,1\}^* \to \{0,1\}^*$ it holds: $\forall \underline{x}$: $\underline{x} \in K \Leftrightarrow f(\underline{x}) \in L$

Lemma: a) $L \in \mathcal{P} \Rightarrow K \in \mathcal{P}$ b) $L \preccurlyeq_p L' \preccurlyeq_p L'' \Rightarrow L \preccurlyeq_p L''$

Travelling Salesperson Problem

HC = { $\langle G \rangle$ | G has Hamiltonian cycle} $\preccurlyeq_p$

TSP = { $\langle G,k \rangle$ | weighted graph G contains cycle of length $\leq k$ }



For $K,L \subseteq \{0,1\}^*$ write $K \preccurlyeq L$ if, for some computable $f: \{0,1\}^* \to \{0,1\}^*$ it holds: $\forall \underline{x}$: $\underline{x} \in K \Leftrightarrow f(\underline{x}) \in L$

$$\mathcal{NP} \ni L = \{ \underline{x} : \exists \underline{y}: |\underline{y}| \leq \mathrm{poly}(|\underline{x}|),\ \langle \underline{x},\underline{y} \rangle \in V \},\quad V \in \mathcal{P}$$

Knapsack

Input: n packets, values $v_1,\ldots v_n \in \mathbb{N}$
and weights $w_1,\ldots w_n \in \mathbb{N}$
and value bound V

Goal: find subset $S \subseteq \{1,\ldots n\}$

- maximizing values $\sum_{p \in S} v_p$
  subject to weight bound $\sum_{p \in S} w_p \leq W$
- minimizing weight $\sum_{p \in S} w_p$
  subject to value bound $\sum_{p \in S} v_p \geq V$

Question: Is there a subset
$S \subseteq \{1,\ldots n\}$ s.t. values $\sum_{p \in S} v_p \geq V$
subject to weight bound $\sum_{p \in S} w_p \leq W$

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